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SONAR ECHO ANALYSIS.(U)

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Department of Mathematics

UNIVERSITY OF UTAH

Salt Lake City, Utah
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6 SONAR ECHO ANALYSIS

10 Calvin H. Wilcox

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ABSTRACT

Pulse mode sonar operation is analyzed under the assumption that the scattering object Γ lies in the far fields of both the transmitter and the receiver. It is shown that, in this approximation, the sonar signal is a plane wave $s(x \cdot \theta_0 - t)$ near Γ, where θ_0 is a unit vector directed from the transmitter toward Γ, and similarly the echo is a plane wave $e(x \cdot \theta - t)$ near the receiver, where θ is a unit vector directed from Γ toward the receiver. Moreover, if Γ is stationary with respect to the sonar system then it is shown that

$$e(\tau) = \mathrm{Re} \left\{ \int_0^\infty e^{i\tau\omega} \, \hat{s}(\omega) \, T_+(\omega\theta, \omega\theta_0) \, d\omega \right\}$$

where $\hat{s}(\omega)$ is the Fourier transform of $s(\tau)$ and $T_+(\omega\theta, \omega\theta_0)$ is the scattering amplitude in the direction θ due to the scattering by Γ of a time-harmonic plane wave with frequency ω and propagation direction θ_0. A generalization of this relation is derived for moving scatterers.

CONTENTS

SECTION ONE

INTRODUCTION - SONAR ECHO PREDICTION

Sonar systems for locating and identifying objects in a fluid environment are used by men and animals of many species including bats, birds, dolphins and other sea mammals, and fish [1]. Some species of bats are known to employ sophisticated frequency-modulated signals and to be able to discriminate accurately among similar objects solely by means of their echoes [1]. Human sonar systems are not yet this effective.

An active sonar system consists of a sound source or "transmitter" and a sound detector or "receiver." The transmitter radiates an acoustic signal which is scattered by objects in its environment. The resulting echoes are recorded by the receiver. In the simplest sonar systems the directions of arrival, time delays and Doppler shifts of the echoes are used to estimate the directions, ranges and speeds of the scattering objects. The goal of a sophisticated system is to classify objects into identifiable classes by means of their echoes. This is an inverse scattering problem. An associated problem is that of waveform design; that is, the choice of a signal waveform which optimizes the echoes from prescribed classes of objects. Before either of these problems can be attacked the direct problem of echo prediction when the signal and scattering object are known must be understood. The purpose of this article is to present a solution to this echo prediction problem. The solution provides a first step toward the understanding of the remarkable efficacy of animal sonar systems which could lead to the design of more effective man-made systems.

Sonar systems are normally operated in the pulse mode in which a sequence of equally spaced short pulses is emitted. A second mode of operation is the CW (continuous wave) mode in which a steady tone of fixed frequency is emitted. The construction of pulse mode echoes given below is derived from the theory of CW mode echoes. The analysis of each of these modes is based on a boundary value problem for an acoustic potential function.

Sonar echo prediction is analyzed below under the following physical assumptions.

- The sonar system (transmitter and receiver) operates in a stationary homogeneous unlimited fluid medium.
- The system is stationary with respect to the medium.
- The sonar signals are generated by conservative force fields.
- The scattering objects are rigid bodies.
- The transmitter and receiver are in the far field of the scattering objects.
- The velocities of the scattering objects are less than that of sound and are essentially constant during the interval required for the sonar pulse to sweep over the object.
- Secondary echoes due to the sonar system components are negligible.
- Noise in the medium is negligible.

The case of a scattering object which is stationary with respect to the sonar system is analyzed first. The analysis is based on the usual linear theory of acoustics [7]. The following notation is used. $x = (x_1, x_2, x_3) \in R^3$ denotes spatial coordinates of a Cartesian system fixed in the medium. $t \in R$ denotes a time coordinate. $\Omega \subset R^3$ denotes the domain exterior to the scattering object and $\Gamma = R^3 - \Omega$ denotes the object. The common frontier of

Γ and Ω, which describes the surface of the object, is denoted by $\partial\Gamma$ or $\partial\Omega$. The equilibrium state of the medium is characterized by its constant density ρ_0, sound speed c_0 and pressure p_0. It will be assumed for notational simplicity that $\rho_0 = 1$ and $c_0 = 1$ since this can be achieved by a suitable choice of units.

The acoustic field generated by the transmitter is characterized by a real-valued acoustic potential function

$$u = u(t,x), \quad t \in R, \ x \in \Omega \tag{1.1}$$

which satisfies the inhomogeneous d'Alembert equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f(t,x) \text{ for } t \in R, \ x \in \Omega \tag{1.2}$$

Here Δ is the Laplacian and $f(t,x)$ is a function, characteristic of the transmitter, which will be called the source function. It has the structure

$$f(t,x) = \partial V(t,x)/\partial t + q(t,x) \tag{1.3}$$

where $V(t,x)$ is a potential for the conservative forces acting in the transmitter and $q(t,x)$ is the volume flow in the transmitter [7, p. 280]. In what follows $f(t,x)$ is assumed to be known.

Pulse Mode Scattering. It will suffice to analyze the scattering of a single short pulse of duration T which is emitted by a transmitter that is localized near a point x_0. Hence the support of $f(t,x)$ will be assumed to satisfy

$$\text{supp } f \subset \{(t,x): \ t_0 \le t \le t_0 + T \text{ and } |x - x_0| \le \delta_0\} \tag{1.4}$$

where δ_0 and t_0 are suitable constants and $|x - x_0|$ is the distance between

x and x_0. The acoustic field generated by f(t,x) is characterized by a potential u(t,x) which satisfies (1.2), the Neumann boundary condition corresponding to a rigid object

$$\frac{\partial u}{\partial \nu} = \nabla u \cdot \vec{\nu} = 0 \text{ for } t \in R,\ x \in \partial\Omega \tag{1.5}$$

where $\vec{\nu} = \vec{\nu}(x)$ is a normal vector to $\partial\Omega$, and the initial condition

$$u(t,x) = 0 \text{ for } t < t_0,\ x \in \Omega. \tag{1.6}$$

The primary field of the transmitter is the field $u_0(t,x)$ generated by f(t,x) when no scattering object is present. It is given by the retarded potential

$$u_0(t,x) = \frac{1}{4\pi} \int_{|x'-x_0| \leq \delta_0} \frac{f(t-|x-x'|,x')}{|x-x'|}\, dx' \text{ for } t \in R,\ x \in R^3 \tag{1.7}$$

where $dx' = dx_1'\, dx_2'\, dx_3'$. The variation of $u_0(t,x)$ near the scatterer Γ may be simplified by using the assumption that the transmitter lies in the far field of Γ. It will be convenient to assume that the origin of coordinates lies in Γ and to let $\delta > 0$ be the smallest constant such that

$$\Gamma \subset \{x: \ |x| \leq \delta\} \tag{1.8}$$

Then the far field assumption takes the form

$$|x_0| >> \delta_0 + \delta \tag{1.9}$$

If θ_0 is the unit vector defined by

$$x_0 = -|x_0|\theta_0 \tag{1.10}$$

and if $|x - x'|$ is developed in inverse powers of $|x_0|$, it is found that

$$|x - x'| = |x_0 + (x' - x_0 - x)|$$

$$= (|x_0|^2 + 2x_0 \cdot (x' - x_0 - x) + |x' - x_0 - x|^2)^{1/2}$$

(1.11)
$$= |x_0| \, (1 - 2\theta_0 \cdot (x' - x_0 - x)/|x_0| + |x' - x_0 - x|^2/|x_0|^2)^{1/2}$$

$$= |x_0| \, (1 - \theta_0 \cdot (x' - x_0 - x)/|x_0| + \mathcal{O}(1/|x_0|^2))$$

$$= |x_0| + \theta_0 \cdot x - \theta_0 \cdot (x' - x_0) + \mathcal{O}(1/|x_0|), \ |x_0| \to \infty$$

and the error term is uniformly small for all $|x' - x_0| \le \delta_0$ and $|x| \le \delta$. Hence, (1.11) may be substituted in (1.7) and gives

(1.12)
$$u_0(t,x) = \frac{s(x\cdot\theta_0 - t + |x_0|)}{|x_0|} + \mathcal{O}(1/|x_0|^2), \ |x_0| \to \infty$$

uniformly for $t \in R$, $|x| \le \delta$ where

(1.13)
$$s(\tau) = \frac{1}{4\pi} \int_{|x'-x_0|\le\delta_0} f(\theta_0 \cdot (x' - x_0) - \tau, x')dx' \text{ for } \tau \in R$$

If the error term in (1.12) is dropped the primary field becomes a plane wave pulse propagating in the direction θ_0. The profile $s(\tau)$ will be called the signal waveform. Note that (1.4) implies supp $s \subset [t_0 - \delta_0 - T, \ t_0 + \delta_0]$.

The acoustic field produced when a plane wave

(1.14)
$$u_0(t,x) = s(x \cdot \theta_0 - t), \text{ supp } s \subset [a,b]$$

is scattered by Γ is the solution $u(t,x)$ of the boundary value problem

(1.15)
$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \text{ for } t \in R, \ x \in \Omega$$

$$(1.16) \qquad \frac{\partial u}{\partial \nu} = 0 \text{ for } t \in R,\ x \in \partial\Omega$$

$$(1.17) \qquad u(t,x) \equiv u_0(t,x) \text{ for } t+b+\delta < 0$$

The secondary field, or echo, is defined by

$$(1.18) \qquad u_{sc}(t,x) = u(t,x) - u_0(t,x) \text{ for } t \in R,\ x \in \Omega$$

It is shown below that in the far field of the scatterer Γ the echo is a diverging spherical wave:

$$(1.19) \qquad u_{sc}(t,x) \sim \frac{e(|x|-t,\theta,\theta_0)}{|x|}\ ,\ x = |x|\theta$$

The profile $e(\tau,\theta,\theta_0)$ will be called the echo waveform. It depends on the direction of incidence θ_0 of the plane wave (1.14) and the direction of observation θ. The principal goal of this article is to calculate the relationship between the echo waveform and the signal waveform. The calculation will be based on the theory of

<u>CW Mode Scattering</u>. CW mode fields are generated by source functions of the form

$$(1.20) \qquad f(t,x) = g_1(x) \cos \omega t + g_2(x) \sin \omega t = \text{Re}\ \{g(x)e^{-i\omega t}\}$$

where $\omega > 0$ is a fixed frequency and $g = g_1 + ig_2$. The corresponding CW mode field has the same time-dependence:

$$(1.21) \qquad u(t,x) = w_1(x) \cos \omega t + w_2(x) \sin \omega t = \text{Re}\ \{w(x)e^{-i\omega t}\}$$

where $w = w_1 + iw_2$. $u(t,x)$ must satisfy the d'Alembert equation (1.2) with $f(t,x)$ defined by (1.20) and the Neumann boundary condition (1.5). The

initial condition (1.6) is not appropriate for CW mode fields and is replaced by the Sommerfeld radiation condition which guarantees that u is a pure outgoing wave in the far field. The corresponding boundary value problem for the complex-valued wave function w(x) is

$$(1.22) \qquad \Delta w + \omega^2 w = -g(x) \text{ for } x \in \Omega$$

$$(1.23) \qquad \frac{\partial w}{\partial \nu} = 0 \text{ for } x \in \partial\Omega$$

$$(1.24) \qquad \frac{\partial w}{\partial |x|} - i\omega w = \mathcal{O}(1/|x|^2) \text{ for } |x| \to \infty$$

It is assumed that

$$(1.25) \qquad \operatorname{supp} g \subset \{x: \ |x - x_0| \le \delta_0\}$$

in agreement with (1.4).

The primary CW mode field $w_0(x)$ is the CW mode field generated by g(x) when no scattering object is present. It is given by

$$(1.26) \qquad w_0(x) = \frac{1}{4\pi} \int_{|x'-x_0|\le\delta_0} \frac{e^{i\omega|x-x'|}}{|x-x'|} g(x')dx' \text{ for } x \in R^3$$

The variation of $w_0(x)$ near Γ may be simplified by means of the far field assumption, as in the case of the pulse mode fields. Substituting (1.11) in (1.26) gives

$$(1.27) \qquad w_0(x) = \frac{T(\omega\theta_0)}{|x_0|} e^{i\omega\theta_0\cdot x} + \mathcal{O}(1/|x_0|^2), \ |x_0| \to \infty$$

uniformly for $|x| \le \delta$ where

$$T(\omega\theta_0) = \frac{1}{4\pi} \int_{|x'-x_0| \le \delta_0} e^{-i\omega\theta_0 \cdot x'} g(x')dx' \tag{1.28}$$

If the error term in (1.27) is dropped the primary field becomes a CW mode plane wave of frequency ω propagating in the direction θ_0. It will be convenient to renormalize the primary field to

$$w_0(x,\omega\theta_0) = (2\pi)^{-3/2} e^{i\omega\theta_0 \cdot x} \tag{1.29}$$

The CW mode field which is produced when $w_0(x,\omega\theta_0)$ is scattered by Γ will be denoted by $w^+(x,\omega\theta_0)$. It is the solution

$$w^+(x,\omega\theta_0) = w_0(x,\omega\theta_0) + w_{sc}^+(x,\omega\theta_0),\ x \in \Omega \tag{1.30}$$

of the boundary value problem

$$\Delta w^+ + \omega^2 w^+ = 0 \text{ for } x \in \Omega \tag{1.31}$$

$$\frac{\partial w^+}{\partial \nu} = 0 \text{ for } x \in \partial\Omega \tag{1.32}$$

$$\frac{\partial w_{sc}^+}{\partial |x|} - i\omega w_{sc}^+ = \mathcal{O}(1/|x|^2),\ |x| \to \infty \tag{1.33}$$

It is shown below that in the far field of Γ the secondary field $w_{sc}^+(x,\omega\theta_0)$ is a diverging spherical wave

$$w_{sc}^+(x,\omega\theta_0) \sim \frac{e^{i\omega|x|}}{4\pi|x|} T_+(\omega\theta,\omega\theta_0),\ x = |x|\theta \tag{1.34}$$

The coefficient $T_+(\omega\theta,\omega\theta_0)$ is called the scattering amplitude of Γ. It determines the amplitude and phase of the CW mode echo in the direction θ due to the primary wave (1.29) with frequency ω and propagation direction θ_0.

The solution of the echo prediction problem for stationary scatterers can now be formulated. It is given by the integral relation

$$(1.35) \qquad e(\tau,\theta,\theta_0) = \mathrm{Re}\left\{\int_0^\infty e^{i\tau\omega}\,\hat{s}(\omega)\,T_+(\omega\theta,\omega\theta_0)\,d\omega\right\}$$

where $\hat{s}(\omega)$ is the Fourier transform of the signal waveform:

$$(1.36) \qquad \hat{s}(\omega) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{-i\omega\tau}\,s(\tau)\,d\tau$$

The derivation of this relation is the principal result of this article.

It is clear that (1.35) is not valid for moving scatterers because it predicts that a signal and its echo contain the same frequencies, while the Doppler shift in echoes from moving scatterers is well known. However, (1.35) can be generalized to the case of a moving scatterer by first applying it in a coordinate frame which is fixed with respect to the scatterer and then passing by means of a Lorentz transformation to a frame which is fixed with respect to the sonar system. This calculation is carried out in §5 below.

Relation (1.35) reveals the fundamental importance for sonar echo analysis of the scattering amplitude $T_+(\omega\theta,\omega\theta_0)$. One simple consequence of (1.35) is the result that the echoes of high frequency pulses are undistorted. Indeed, it has recently been shown by A. Majda [3] that

$$(1.37) \qquad \lim_{\omega\to\infty} T_+(\omega\theta,\omega\theta_0) = T_+^\infty(\theta,\theta_0)$$

exists when Γ is a smooth convex object. It follows from (1.35) that if $\hat{s}(\omega)$ is concentrated in a high frequency band $\omega \geq \omega_0$ where $T_+(\omega\theta,\omega\theta_0)$ is nearly constant then

$$(1.38)\qquad e(\tau,\theta,\theta_0) \sim T_+(\theta,\theta_0)\,\mathrm{Re}\,\{\int_0^\infty e^{i\tau\omega}\,\hat{s}(\omega)d\omega\} = \left(\frac{\pi}{2}\right)^{1/2} T_+^\infty(\theta,\theta_0)s(\tau)$$

by Fourier's theorem. More interesting, however is the possibility of applying (1.35) to the study of pulse distortion by reflection and its dependence on the geometry of the scatterer. In particular, (1.35) shows that resonances in $T_+(\omega\theta,\omega\theta_0)$ will produce selective enhancement of sonar echoes at the resonance frequencies. Another possible application of (1.35) is to use Fourier analysis of observed sonar echoes produced by known signals to estimate the function $T_+(\omega\theta,\omega\theta_0)$ over a range of frequencies. These studies are clearly applicable to the analysis and design of sonar systems.

This article is based on the author's monograph [10] to which reference is made for analytical details. The theory of scattering developed in [10] is applicable to a large class of objects Γ with irregular, non-smooth boundaries. The class is defined by a "local compactness property" of the domains $\Omega = R^3 - \Gamma$ [10, p. 60]. A rather general geometric criterion for the validity of this property is given in [10, p. 63]. The class of allowable objects Γ includes all the simple objects that arise in applications such as polyhedra, finite sections of cylinders, cones, spheres, disks and, more generally, all objects with piecewise smooth surfaces which have a finite number of smooth edges and vertices. It is assumed throughout this article that Ω has the local compactness property so that the results of [10] are applicable.

The remainder of the article is organized as follows. §2 reviews briefly the definition and structure of CW mode wave fields. §3 reviews the definition of pulse mode sonar wave fields and their representation by means

of CW mode fields. §4 describes the theory of asymptotic wave fields, as developed in [10], and applies it to the derivation of the far field form (1.19) of sonar echoes and the integral relation (1.35). §5 contains the generalization of (1.35) to the case of moving scatterers.

SECTION TWO

THE STRUCTURE OF CW MODE SONAR ECHOES

The definition and basic properties of the CW mode fields $w^+(x,p)$ are reviewed in this section. The definition is based on the boundary value problem (1.30) - (1.33) in a formulation appropriate for domains with non-smooth boundaries. In the classical theory due to Kupradze [2] and Weyl [8] a solution is a function in the class $C^2(\Omega) \cap C^1(\overline{\Omega})$ where $\overline{\Omega} = \Omega \cup \partial\Omega$. It is known that if $\partial\Omega$ is smooth then a unique classical solution $w^+(x,p)$ exists for each $p = \omega\theta_0 \in R^3$. However, if $\partial\Omega$ has edges and/or vertices then at such points the normal vector $\vec{\nu}$ is undefined and the boundary condition (1.32) is not meaningful. In this case it was discovered that if (1.32) is enforced only at points where $\partial\Omega$ is smooth then all solutions have first derivatives which are singular at the edges and vertices and no classical solution exists [4, 5]. Moreover, if the strength of these singularities is not controlled then the uniqueness of the solution is lost [5]. Physically, non-uniqueness can occur because line or point sources can be situated in the edges or vertices. This type of non-uniqueness is eliminated by the "edge condition" [5] which requires that the energy in each bounded portion of Ω should be finite. It may be formulated mathematically as the condition that $u(x) = w^+(x,\omega\theta_0)$ should satisfy [5]

$$(2.1) \qquad \int_{K\cap\Omega} \{|\nabla u(x)|^2 + \omega^2|u(x)|^2\}dx < \infty \text{ for every cube } K \subset R^3$$

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In the theory developed in [10] the edge condition (2.1) is formulated by means of the function classes

$$(2.2) \qquad L_2^{\ell oc}(\overline{\Omega}) = \{u: \ u \in L_2(K \cap \Omega) \text{ for every cube } K \subset R^3\}$$

$$(2.3) \qquad L_2^{1,\ell oc}(\overline{\Omega}) = L_2^{\ell oc}(\overline{\Omega}) \cap \{u: \ \frac{\partial u}{\partial x_j} \in L_2^{\ell oc}(\overline{\Omega}) \text{ for } j = 1,2,3\}$$

where $L_2(M)$ denotes the usual Lebesgue class of a set $M \subset R^3$. Condition (2.1) is equivalent to the condition $u \in L_2^{1,\ell oc}(\overline{\Omega})$. Moreover, if $u = w^+ \in L_2^{1,\ell oc}(\overline{\Omega})$ is a weak solution of (1.31), in the sense of the theory of distributions, then u is also in the class

$$(2.4) \qquad L_2^{1,\ell oc}(\Delta,\overline{\Omega}) = L_2^{1,\ell oc}(\overline{\Omega}) \cap \{u: \ \Delta u \in L_2^{\ell oc}(\overline{\Omega})\}$$

For general domains $\Omega \subset R^3$ and functions $u \in L_2^{1,\ell oc}(\Delta,\overline{\Omega})$ the Neumann condition (1.32) is replaced by the generalized Neumann condition of [10]:

$$(2.5) \qquad \int_\Omega \{(\Delta u)v + \nabla u \cdot \nabla v\} dx = 0$$

for all $v \in L_2(\Omega)$ such that $\nabla v \in L_2(\Omega)$ and $v(x) = 0$ outside of a bounded set. The terminology is justified by the fact that (2.5) implies that the classical Neumann condition $\partial u(x)/\partial \nu = 0$ is satisfied at every boundary point x where $\partial\Omega$ is smooth [10, p. 41].

The CW mode field $w^+(x,p)$ corresponding to a primary field

$$(2.6) \qquad w_0(x,p) = (2\pi)^{-3/2} \ e^{ip\cdot x}, \ p \in R^3$$

and an exterior domain $\Omega \subset R^3$ is defined in [10] by the conditions

$$(2.7) \qquad w^+(\cdot,p) \in L_2^{N,\ell oc}(\Delta,\overline{\Omega}) = L_2^{1,\ell oc}(\Delta,\overline{\Omega}) \cap \{u: \ u \text{ satisfies } (2.5)\}$$

$$(\Delta + |p|^2)\, w^+(x,p) = 0 \text{ for } x \in \Omega \tag{2.8}$$

$$\frac{\partial w^+_{sc}}{\partial |x|} - i|p|\, w^+_{sc} = O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty, \text{ where } w^+_{sc} = w^+ - w_0 \tag{2.9}$$

In this formulation the edge condition and the generalized Neumann condition are incorporated into condition (2.7). Moreover, since it is known that weak solutions of (2.8) are necessarily analytic functions of $x \in \Omega$ [6], (2.9) is meaningful for functions which satisfy (2.7) and (2.8). The uniqueness of the functions $w^+(x,p)$ was proved in [10] for arbitrary exterior domains and their existence was proved for domains Ω which have the local compactness property.

The remainder of this section presents a calculation of the far field form (1.34) of the CW echo $w^+_{sc}(x,p)$. It is based on the integral representation [10, p. 90]

$$w^+_{sc}(x,p) = \int_{|x'|=r_0} \left\{ w^+(x',p) \frac{\partial G_0(x-x',p)}{\partial |x'|} - \frac{\partial w^+(x',p)}{\partial |x'|} G_0(x-x',p) \right\} dS', \quad |x| > r_0 \tag{2.10}$$

where $r_0 > \delta$ and

$$G_0(x-x',p) = \frac{e^{i|p||x-x'|}}{4\pi|x-x'|} \tag{2.11}$$

The asymptotic form of $w^+_{sc}(x,p)$ for $x = |x|\theta$, $|x| \to \infty$ may be calculated from (2.10) by using the approximation $|x-x'| = |x| - x' \cdot \theta + O(1/|x|)$ as in §1. The result is

$$w_{sc}^{+}(x,p) = \frac{e^{i|p||x|}}{4\pi|x|} T_{+}(|p|\theta,p) + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty \tag{2.12}$$

uniformly for all directions θ and p in any bounded set, where the cross section $T_{+}(p,p')$ may be written

$$T_{+}(p,p') = \int_{|x|=r_0} \{w^{+}(x,p') \frac{\partial}{\partial|x|} e^{-ip\cdot x} - e^{-ip\cdot x} \frac{\partial w^{+}(x,p')}{\partial|x|}\} dS \tag{2.13}$$

It follows from Green's theorem that the integral in (2.13) is independent of r_0 provided that $r_0 > \delta$ and $|p'| = |p|$.

The well known reciprocity theorem for $T_{+}(p,p')$ states that

$$T_{+}(p,p') = T_{+}(-p',-p) \tag{2.14}$$

for all p and p' such that $|p| = |p'|$. This relation may be verified in the present setting by applying Green's theorem to $w^{+}(x,p)$ and $w^{+}(x,p')$ in the region $\Omega r_0 = \Omega \cap \{x: \ |x| < r_0\}$ and using the representation (2.13). If $\partial\Omega$ is not smooth the calculation must be based on the version of Green's theorem given by the generalized Neumann condition (2.5). The technique for doing this is described in [10, pp. 57-8].

It can be shown that if the scatterer Γ has a piecewise smooth surface $\partial\Gamma = \partial\Omega$ then $w^{+}(x,p)$ is continuous and $\partial w^{+}(x,p)/\partial\nu = 0$ at all boundary points except edges and vertices. In this case Green's theorem can be applied to $w_0(x,-p)$ and $w^{+}(x,p')$ in $\Omega(r_0)$. This calculation, together with (2.13), implies the representation

$$T_{+}(p,p') = \int_{\partial\Omega} w^{+}(x,p') \frac{\partial}{\partial\nu} (e^{-ip\cdot x}) dS \tag{2.15}$$

Moreover, under the same hypothesis the application of Green's theorem to

$G_0(x - x', p)$ and $w^+(x', p)$ gives the representation

$$(2.16) \qquad w^+(x,p) = w_0(x,p) + \int_{\partial\Omega} \frac{\partial G_0(x-x',p)}{\partial \nu'} w^+(x',p)\,dS', \quad x \in \Omega$$

and the unknown boundary values of $w^+(x,p)$ satisfy the Fredholm equation [4]

$$(2.17) \qquad w^+(x,p) = 2w_0(x,p) + 2\int_{\partial\Omega} \frac{\partial G_0(x-x',p)}{\partial \nu'} w^+(x',p)\,dS', \quad x \in \partial\Omega$$

These relations provide a basis for both exact and approximate calculations of $w^+(x,p)$ and $T_+(p,p')$.

SECTION THREE

THE STRUCTURE OF PULSE MODE SONAR ECHOES

In this section the Hilbert space method of [10] is used to construct the sonar echoes produced by pulse mode signals. The construction is based on the fact that the negative Laplacian $A = -\Delta$, acting on functions that satisfy the edge condition and generalized Neumann condition, defines a selfadjoint operator in the Hilbert space $L_2(\Omega)$. It was shown in [10] that the CW mode fields $w^+(x,p)$ form a complete set of generalized eigenfunctions for A. This result is used below to derive an integral representation of pulse mode sonar echoes.

The edge condition and generalized Neumann condition for function in $L_2(\Omega)$ are formulated in [10] by means of the function classes

$$(3.1) \qquad L_2^1(\Omega) = L_2(\Omega) \cap \{u: \frac{\partial u}{\partial x_j} \in L_2(\Omega) \text{ for } j = 1,2,3\}$$

$$(3.2) \qquad L_2^1(\Delta,\Omega) = L_2^1(\Omega) \cap \{u: \Delta u \in L_2(\Omega)\}$$

$$(3.3) \qquad L_2^N(\Delta,\Omega) = L_2^1(\Delta,\Omega) \cap \{u: u \text{ satisfies (2.5) for all } v \in L_2^1(\Omega)\}$$

The operator A in $L_2(\Omega)$ is defined by

$$(3.4) \qquad D(A) = L_2^N(\Delta,\Omega)$$

$$(3.5) \qquad Au = -\Delta u \text{ for all } u \in D(A)$$

It was shown in [10, p. 41] that A is a selfadjoint non-negative operator in $L_2(\Omega)$:

(3.6) $$A = A^* \geq 0$$

The non-negative square root of A, denoted by $A^{1/2}$, is also needed below. It was shown in [10] that

(3.7) $$D(A^{1/2}) = L_2^1(\Omega)$$

(3.8) $$\|A^{1/2}u\| = \|\nabla u\| \quad \text{for all } u \in D(A^{1/2})$$

where $\|\cdot\|$ denotes the $L_2(\Omega)$-norm.

The d'Alembert equation (1.2) is interpreted below as an ordinary differential equation

(3.9) $$\frac{d^2u}{dt^2} + Au = f(t,\cdot), \quad t \in R$$

for a function $t \to u(t,\cdot) \in L_2(\Omega)$. It is assumed that

(3.10) $$\operatorname{supp} f \subset \{t\colon \; t_0 \leq t \leq t_0 + T\}$$

and a solution of (3.9) is sought which satisfies

(3.11) $$u(t,\cdot) = 0 \text{ for } t < t_0$$

in agreement with (1.4), (1.6).

A formal solution of (3.9), (3.11) is defined by the Duhamel integral

(3.12) $$u(t,\cdot) = \int_{t_0}^{t} \{A^{-1/2} \sin (t-\tau)A^{1/2}\} f(\tau,\cdot)d\tau, \quad t \geq t_0$$

where $A^{-1/2} \sin t\, A^{1/2}$ is defined by means of the spectral theorem. Integration by parts in (3.12) gives the alternative representation

$$u(t,\cdot) = \int_{t_0}^{t} \{\cos (t-\tau)A^{1/2}\} \int_{t_0}^{\tau} f(t',\cdot)\, dt'\, d\tau \tag{3.13}$$

It will be assumed that the source function f satisfies

$$f \in C(R,L_2^1(\Omega)) = C(R,D(A^{1/2})) \tag{3.14}$$

It can then be verified that (3.12) defines a function

$$u \in C^2(R,L_2(\Omega)) \cap C^1(R,D(A^{1/2})) \cap C(R,D(A)) \tag{3.15}$$

whose first and second t-derivatives satisfy

$$\frac{du}{dt} = \int_{t_0}^{t} \{\cos (t-\tau)A^{1/2}\} f(\tau,\cdot)d\tau \tag{3.16}$$

and

$$\begin{aligned} \frac{d^2u}{dt^2} &= f(t,\cdot) - \int_{t_0}^{t} \{A^{1/2} \sin (t-\tau)A^{1/2}\} f(\tau,\cdot)\, d\tau \\ &= f(t,\cdot) - Au \end{aligned} \tag{3.17}$$

In particular, $u(t,\cdot) \in D(A)$ for every $t \in R$ and hence $u(t,x)$ satisfies the edge condition and generalized Neumann condition for every t. The uniqueness of this kind of solution was proved in [9].

The remainder of this article deals with acoustic fields of the form (3.12) at times $t \geq t_0 + T$; i.e., after the primary field has been established and the sources have ceased to act. It will be assumed that

$$\int_{t_0}^{t_0+T} f(t,x)\, dt = 0 \text{ for all } x \in \Omega \tag{3.18}$$

This hypothesis is made to simplify the analysis and is not essential to the method. The physical meaning of (3.18) is clear from (1.3).

Equations (3.13) and (3.18) imply that

$$u(t,\cdot) = \int_{t_0}^{t_0+T} \{\cos\,(t-\tau)A^{1/2}\} \int_{t_0}^{\tau} f(t',\cdot)dt'd\tau \qquad \text{for } t \geq t_0 + T \tag{3.19}$$

Moreover, since $f(t,x)$ is real-valued (3.19) implies that

$$u(t,x) = \operatorname{Re}\,\{v(t,x)\} \text{ for } t \geq t_0 + T \tag{3.20}$$

where

$$v(t,\cdot) = \int_{t_0}^{t_0+T} e^{-i(t-\tau)A^{1/2}} \int_{t_0}^{\tau} f(t',\cdot)dt'd\tau = e^{-itA^{1/2}}h \tag{3.21}$$

and

$$h = \int_{t_0}^{t_0+T} e^{i\tau A^{1/2}} \int_{t_0}^{\tau} f(t',\cdot)\,dt'd\tau \tag{3.22}$$

These equations contain, in abstract form, the solution of the sonar echo prediction problem. A more concrete representation is provided by the eigenfunction expansion theorem of [10] which will be reviewed here briefly and applied to (3.21).

The expansion theorem states that every $h \in L_2(\Omega)$ has a generalized Fourier transform

$$\hat{h}_+(p) = L_2(R^3)\text{-}\lim_{M\to\infty} \int_{\Omega_M} \overline{w^+(x,p)}\; h(x)\,dx \tag{3.23}$$

where $\Omega_M = \Omega \cap \{x\colon |x| < M\}$ and a corresponding eigenfunction representation

$$h(x) = L_2(\Omega)\text{-}\lim_{M\to\infty} \int_{|p|\leq M} w^+(x,p)\,\hat{h}_+(p)\,dp \tag{3.24}$$

Moreover, the transformation Φ_+: $L_2(\Omega) \to L_2(R^3)$ defined by

$$\Phi_+ h = \hat{h}_+, \quad h \in L_2(\Omega) \tag{3.25}$$

is unitary and diagonalizes A in the sense that

$$(\Phi_+ \Psi(A) h)(p) = \Psi(|p|^2)\,\Phi_+ h(p) \tag{3.26}$$

for every bounded measurable function $\Psi(\lambda)$ which is defined for $\lambda \geq 0$. In what follows the relations (3.23), (3.24) will be written in the symbolic form

$$\hat{h}_+(p) = \int_\Omega \overline{w^+(x,p)}\,h(x)\,dx \tag{3.27}$$

$$h(x) = \int_{R^3} w^+(x,p)\,\hat{h}_+(p)\,dp \tag{3.28}$$

However, these integrals are not in general convergent and must be interpreted in the sense of (3.23), (3.24).

A second complete family of generalized eigenfunctions for A is defined by

$$w^-(x,p) = \overline{w^+(x,-p)} \tag{3.29}$$

This family can also be characterized by the boundary value problem (2.6)-(2.9), but with the outgoing radiation condition (2.9) replaced by the incoming radiation condition

$$\frac{\partial w_{sc}^-}{\partial |x|} + i|p|\,w_{sc}^- = O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty, \text{ where } w_{sc}^- = w^- - w_0 \tag{3.30}$$

The relations (3.23) - (3.26) hold with $w^{+}(x,p)$, $\hat{h}_{+}(p)$ and Φ_{+} replaced by $w^{-}(x,p)$, $\hat{h}_{-}(p)$ and Φ_{-}, respectively. This was proved in [10]. It also follows directly from (3.23) - (3.26) and (3.29). The far field behavior of $w^{-}(x,p)$ follows from (2.12) and (3.29):

$$(3.31) \qquad w^{-}_{sc}(x,p) = \frac{e^{-i|p||x|}}{4\pi|x|} T_{-}(|p|\theta,p) + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty$$

where

$$(3.32) \qquad T_{-}(p,p') = \overline{T_{+}(p,-p')}$$

Application of (3.26) to (3.21) gives

$$(3.33) \qquad (\Phi_{+}\, e^{-itA^{1/2}} h)(p) = e^{-it|p|}\, \hat{h}_{+}(p)$$

and hence, by (3.24)

$$(3.34) \qquad v(t,x) = \int_{R^3} w^{+}(x,p)\, e^{-it|p|}\, \hat{h}_{+}(p)\, dp$$

The family $\{w^{-}(x,p): \ p \in R^3\}$ provides a second representation

$$(3.35) \qquad v(t,x) = \int_{R^3} w^{-}(x,p)\, e^{-it|p|}\, \hat{h}_{-}(p)\, dp$$

where

$$(3.36) \qquad \hat{h}_{-}(p) = \int_{\Omega} \overline{w^{-}(x,p)}\, h(x)dx$$

If h is defined by (3.22) where f satisfies (3.10), (3.14), (3.18) then the continuity of Φ_{+} and Φ_{-}, together with (3.26), imply that

$$\hat{h}_{\pm}(p) = \int_{t_0}^{t_0+T} e^{i\tau|p|} \int_{t_0}^{\tau} \Phi_{\pm}\, f(t,p)\,dt\,d\tau \tag{3.37}$$

Interchange of the t and τ integrations and use of (3.18) gives

$$\hat{h}_{\pm}(p) = i|p|^{-1} \int_{t_0}^{t_0+T} \int_{\Omega} \overline{e^{-i|p|t}\, w^{\pm}(x,p)}\, f(t,x)\,dx\,dt \tag{3.38}$$

or

$$\hat{h}_{\pm}(p) = (2\pi)^{1/2}\, i\, |p|^{-1}\, \hat{f}_{\pm}(-|p|,p) \tag{3.39}$$

where

$$\hat{f}_{\pm}(\omega,p) = \frac{1}{(2\pi)^{1/2}} \int_{t_0}^{t_0+T} \int_{\Omega} \overline{e^{i\omega t}\, w^{\pm}(x,p)}\, f(t,x)\,dx\,dt \tag{3.40}$$

The total field produced by the scattering of acoustic waves with a source function $f(t,x)$ is represented by (3.34) and (3.35) where $\hat{h}_{+}(p)$ and $\hat{h}_{-}(p)$ are defined by (3.39), (3.40). These equations are not directly applicable to the boundary value problem (1.14) - (1.17) for the scattering of plane waves. However, it will be shown that this problem can be reduced to the preceding one. To this end let r_0 and R be radii such that $\delta < r_0 < R$ and let $j(x) \in C^{\infty}(R^3)$ be a function with the properties

$$j(x) = 0 \text{ for } |x| \le r_0,\ j(x) = 1 \text{ for } |x| \ge R \tag{3.41}$$

The solution of the boundary value problem (1.14) - (1.17) will be constructed as a sum

$$u(t,x) = j(x)\, u_0(t,x) + \tilde{u}_{sc}(t,x) \tag{3.42}$$

Note that $\tilde{u}_{sc}(t,x)$ coincides with the echo (1.18) for $|x| \geq R$ and all $t \in R$, by (3.41).

The solution of (1.14) - (1.17) is given by (3.42) if $\tilde{u}_{sc}(t,x)$ satisfies

(3.43) $$\frac{\partial^2 \tilde{u}_{sc}}{\partial t^2} - \Delta \tilde{u}_{sc} = f(t,x) \text{ for } t \in R,\ x \in \Omega$$

(3.44) $$\frac{\partial \tilde{u}_{sc}}{\partial \nu} = 0 \text{ for } t \in R,\ x \in \partial\Omega$$

(3.45) $$\tilde{u}_{sc}(t,x) = 0 \text{ for } t + b + R < 0,\ x \in \Omega$$

where

(3.46) $$f(t,x) = -\left(\frac{\partial^2}{\partial t^2} - \Delta\right) j(x)\ u_0(t,x)$$

Note that

(3.47) $$\text{supp } f \subset \{(t,x):\ -b - R \leq t \leq -a + R \text{ and } r_0 \leq |x| \leq R\}$$

by (1.14) and (3.41). In particular, (3.10) holds with $t_0 = -b - R$ and $T = 2R + b - a$. Moreover, (3.14) holds if $s(\tau) \in L_2^2(R) = L_2(R) \cap \{s(\tau):\ s'(\tau) \text{ and } s''(\tau) \text{ are in } L_2(R)\}$, and (3.18) holds if

(3.48) $$\int_a^b s(\tau)\, d\tau = 0$$

Of course, if $\partial\Gamma$ is not smooth then $\tilde{u}_{sc}$ must satisfy the edge condition and generalized Neumann condition; i.e., $\tilde{u}_{sc}(t,\cdot) \in D(A) = L_2^N(\Delta,\Omega)$ for all $t \in R$. The solution is given by

(3.49) $$\tilde{u}_{sc}(t,x) = \text{Re}\ \{\tilde{v}_{sc}(t,x)\}$$

where $\tilde{v}_{sc}$ has the two representations

$$(3.50) \qquad \tilde{v}_{sc}(t,x) = \int_{R^3} w^{\pm}(x,p)\, e^{-it|p|}\, \hat{h}_{\pm}(p)dp$$

corresponding to w^+ and w^-, and $\hat{h}_+(p)$ and $\hat{h}_-(p)$ are defined by (3.39), (3.40) and (3.46). This section is concluded with a calculation of $\hat{h}_-(p)$.

The support of f in space-time is bounded, by (3.47), and a simple calculation gives

$$(3.51) \qquad \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-i\omega t}\, f(t,x)dt = \hat{s}(-\omega)(\Delta+\omega^2)\, j(x)\, e^{-i\omega\theta_0\cdot x}$$

where $\hat{s}(\omega)$ is defined by (1.36). Thus, by (3.29) and (3.51),

$$(3.52) \qquad \hat{f}_-(\omega,p) = \hat{s}(-\omega) \int_{\Omega} w^+(x,-p)(\Delta+\omega^2)\, j(x)\, e^{-i\omega\theta_0\cdot x}\, dx$$

Now by (3.41)

$$(3.53) \qquad \text{supp } (\Delta+\omega^2)\, j(x)\, e^{-i\omega\theta_0\cdot x} \subset \{x: \ r_0 \le |x| < R\} = \Omega_R - \Omega_{r_0}$$

Hence, replacing Ω by $\Omega_R - \Omega_{r_0}$ in (3.52) and applying Green's theorem to $w^+(x,-p)$ and $j(x)\, e^{i|p|\theta_0\cdot x}$ in the region $r_0 \le |x| \le R$ gives

$$(3.54) \qquad \begin{aligned} \hat{f}_-(-|p|,p) &= \hat{s}(|p|) \int_{S(R)} \{w^+(x,-p) \frac{\partial}{\partial|x|} e^{i|p|\theta_0\cdot x} - e^{i|p|\theta_0\cdot x} \frac{\partial w^+(x,-p)}{\partial|x|}\}\, dS \\ &= \hat{s}(|p|)\, T_+(-|p|\theta_0,-p) = \hat{s}(|p|)T_+(p,|p|\theta_0) \end{aligned}$$

by (2.13) and (2.14). Thus, by (3.39)

$$(3.55) \qquad \hat{h}_-(p) = (2\pi)^{1/2}\, i\, |p|^{-1}\, \hat{s}(|p|)\, T_+(p,|p|\theta_0)$$

Combining (3.49), (3.50) and (3.55) gives the representation

$$(3.56)\qquad u_{sc}(t,x) = \mathrm{Re}\left\langle\left\{(2\pi)^{1/2}\, i \int_{R^3} w^{-}(x,p)e^{-it|p|}|p|^{-1}\hat{s}(|p|)T_{+}(p,|p|\theta_0)dp\right\}\right\rangle$$

for $|x| \geq R$. Actually, (3.56) is valid for all $x \in \Omega$. This can be seen by noting that (3.56) does not depend on $j(x)$ and that condition (3.41) may be replaced by the condition that supp $\nabla j(x)$ is contained in an arbitrarily small neighborhood of Γ.

SECTION FOUR

SONAR ECHOES IN THE FAR FIELD

The sonar echo $u_{sc}(t,x)$ of the primary field $u_0(t,x) = s(x \cdot \theta_0 - t)$ originates at Γ and reaches points x in the far field, characterized by $|x| >> \delta$, after a time interval of magnitude comparable with $|x|$. Hence the far field form of $u_{sc}(t,x)$ coincides with its asymptotic form for large t. The latter is provided by the theory of asymptotic wave functions developed in [10]. The relevant parts of the theory are summarized here and applied to $u_{sc}(t,x)$.

The simplest theorem concerning asymptotic wave functions states that if

$$u(t,x) = \text{Re}\,\{v(t,x)\},\quad v(t,\cdot) = e^{-itA^{1/2}} h,\quad h \in L_2(\Omega) \tag{4.1}$$

then

$$\lim_{t\to\infty} \| u(t,\cdot) - u^{\infty}(t,\cdot)\| = 0 \tag{4.2}$$

where $u^{\infty}(t,x)$, the asymptotic wave function for $u(t,x)$, is defined by

$$u^{\infty}(t,x) = \frac{F(|x|-t,\theta)}{|x|},\quad x = |x|\theta \tag{4.3}$$

and

$$F(\tau,\theta) = \text{Re}\left\{\frac{1}{(2\pi)^{1/2}} \int_0^{\infty} e^{i\tau\omega}\, \hat{h}_{-}(\omega\theta)(-i\omega)\,d\omega\right\} \tag{4.4}$$

where $\hat{h}_{-}(p)$ is defined by (3.36). This result is proved in [10], Corollary 8.3.

Equations (4.1) - (4.4) can be applied to the echo $u_{sc}(t,x)$. Indeed, (3.56) implies that

(4.5) $$u_{sc}(t,x) = \mathrm{Re}\,\{v_{sc}(t,x)\},\quad v_{sc}(t,\cdot) = e^{-itA^{1/2}}\,h$$

where h is defined by (3.55). The calculation leading to (3.55) shows that h is also given by

(4.6) $$h(x) = \int_{t_0}^{t_0+T} e^{i\tau A^{1/2}} \int_{t_0}^{\tau} f(t,x)\,dt\,d\tau$$

where $f(t,x)$ is defined by (3.46). A simple calculation based on (3.46) and (4.6) gives the alternative representation

(4.7) $$h(x) = \int_{t_0}^{t_0+T} e^{i\tau A^{1/2}} \left\{\Delta j(x) \int_{x\cdot\theta_0-\tau}^{\infty} s(t)\,dt - \theta_0\cdot\nabla j(x)\; s(x\cdot\theta_0-\tau)\right\} d\tau$$

It follows that if

(4.8) $$s \in L_2^2(R),\quad \mathrm{supp}\; s \subset [a,b],\quad \int_a^b s(\tau)\,d\tau = 0$$

as was assumed above, then $h \in D(A)$. In particular, $h \in L_2(\Omega)$ and $u_{sc}(t,x)$ has the asymptotic wave function

(4.9) $$u_{sc}^{\infty}(t,x) = \frac{e(|x|-t,\theta,\theta_0)}{|x|},\quad x = |x|\theta$$

where $e(\tau,\theta,\theta_0) = F(\tau,\theta)$ is given by (4.4) with $\hat{h}_-$ defined by (3.55). Substitution gives equation (1.35) of §1; i.e.,

(4.10) $$e(\tau,\theta,\theta_0) = \mathrm{Re}\left\{\int_0^{\infty} e^{i\tau\omega}\,\hat{s}(\omega)\;T_+(\omega\theta,\omega\theta_0)\,d\omega\right\}$$

Moreover, (1.19) of §1 now has the interpretation

$$\lim_{t\to\infty} \| u_{sc}(t,\cdot) - u_{sc}^{\infty}(t,\cdot)\| = 0 \tag{4.11}$$

Stronger convergence results follow from the hypotheses (4.8). Indeed, (4.8) implies that $h \in D(A^{1/2}) = L_2^1(\Omega)$ and hence Theorem 8.5 of [10] implies that

$$\lim_{t\to\infty} \left\| \frac{\partial u_{sc}(t,\cdot)}{\partial x_j} - u_{sc,j}^{\infty}(t,\cdot) \right\| = 0, \quad j = 0,1,2,3 \tag{4.12}$$

where $x_0 = t$,

$$u_{sc,j}^{\infty}(t,x) = \frac{e_j(|x|-t,\theta,\theta_0)}{|x|}, \quad j = 0,1,2,3 \tag{4.13}$$

$$\begin{aligned} e_0(\tau,\theta,\theta_0) &= -\frac{\partial e(\tau,\theta,\theta_0)}{\partial \tau} \\ &= \mathrm{Re}\,\{\int_0^{\infty} e^{i\tau\omega}\,(-i\omega)\,\hat{s}(\omega)\,T_+(\omega\theta,\omega\theta_0)\,d\omega\} \end{aligned} \tag{4.14}$$

and

$$e_j(\tau,\theta,\theta_0) = -\theta_j\, e_0(\tau,\theta,\theta_0) \text{ for } j = 1,2,3 \tag{4.15}$$

Note that

$$u_{sc,0}^{\infty}(t,x) = \frac{\partial u_{sc}^{\infty}(t,x)}{\partial t} \tag{4.16}$$

The remaining asymptotic wave functions, corresponding to the space derivatives, are not the derivatives of u_{sc}^{∞} but differ from them by terms which converge to zero in $L_2(\Omega)$ when $t \to \infty$. This can be proved by means of Lemma 2.7 of [10].

SECTION FIVE

SONAR ECHOES FROM MOVING OBJECTS

The preceding sections treated scattering from objects that were stationary with respect to the sonar system. In this section the analysis is extended to moving scatterers under the simplifying assumption that their velocities are subsonic and essentially constant during the scattering of a pulse.

The relative velocity of the sonar system and the scatterer during the scattering will be described by a constant vector $\vec{v}$ with magnitude $v = |\vec{v}|$ such that $0 < v < 1$ (= speed of sound). The space-time coordinates of a Galilean reference frame in which the sonar system is at rest will be denoted by $(t,x) = (t,x_1,x_2,x_3)$, as in the preceding sections. The coordinates of a second Galilean frame in which the scatterer Γ is at rest will be denoted by $(t',x') = (t',x_1',x_2',x_3')$. The two coordinate systems are related by a Lorentz transformation based on $\vec{v}$. It may be assumed that the spatial axes of the two systems are parallel and that the origin $0'$ of the second system moves along the x_1-axis in the positive direction with speed v. The Lorentz transformation then takes the form

$$(5.1) \qquad t' = \frac{t-vx_1}{\sqrt{1-v^2}}, \quad x_1' = \frac{x_1-vt}{\sqrt{1-v^2}}, \quad x_2' = x_2, \quad x_3' = x_3$$

with inverse

(5.2) $$t = \frac{t'+vx_1'}{\sqrt{1-v^2}}, \quad x_1 = \frac{x_1'+vt'}{\sqrt{1-v^2}}, \quad x_2 = x_2', \quad x_3 = x_3'$$

The assumption that Γ is in the far field of the transmitter implies that the sonar signal is effectively a plane wave near Γ, as was shown in §1. Thus the signal is described relative to the sonar system frame by a scalar potential

(5.3) $$u_0(t,x) = s(x \cdot \theta_0 - t)$$

The Lorentz transformation (5.1) satisfies the relation

(5.4) $$x \cdot \theta_0 - t = \gamma_0(x' \cdot \theta_0' - t')$$

where

(5.5) $$\gamma_0 = \frac{1-v\theta_0^1}{\sqrt{1-v^2}} = \frac{1-\vec{v}\cdot\theta_0}{\sqrt{1-v^2}}$$

and

(5.6) $$\theta_0'^{1} = \frac{\theta_0^1-v}{1-v\theta_0^1}, \quad \theta_0'^{2} = \frac{\sqrt{1-v^2}\,\theta_0^2}{1-v\theta_0^1}, \quad \theta_0'^{3} = \frac{\sqrt{1-v^2}\,\theta_0^3}{1-v\theta_0^1}$$

Thus the signal takes the form

(5.7) $$u_0'(t',x') = s'(x' \cdot \theta_0' - t')$$

relative to the object frame where

(5.8) $$s'(\tau) = s(\gamma_0\tau)$$

It is easy to verify that (5.6) defines a mapping of the unit vectors $\theta_0 \in R^3$ into unit vectors $\theta_0' \in R^3$.

The d'Alembert equation (1.15) is invariant under the Lorentz transformation (5.1). Hence, relative to the object frame coordinates (t',x') the echo produced by the scattering of the signal (5.7) from Γ is the solution of the same boundary value problem (1.15) - (1.18) that was solved in §3. It follows from the analysis of §4 that the echo is given in the far field by the asymptotic wave function

$$u_{sc}^{\infty\prime}(t',x') = \frac{e'(|x'|-t',\theta',\theta_0')}{|x'|}, \quad x' = |x'|\theta' \tag{5.9}$$

with echo waveform

$$e'(\tau,\theta',\theta_0') = \text{Re}\left\{\int_0^\infty e^{i\tau\omega}\, \hat{s}'(\omega)\, T_+(\omega\theta',\omega\theta_0')\, d\omega\right\} \tag{5.10}$$

where $T_+(p,p')$ is the same function defined in §2.

Equations (5.9), (5.10) describe the echo relative to the object frame. To describe it relative to the sonar system frame it is necessary to apply the Lorentz transformation again. This may be done by noting that along the ray $x' = |x'|\theta'$ with θ' fixed and $|x'| \geq 0$ one has $|x'| = x'\cdot\theta'$. Hence, using (5.4) - (5.6) with θ_0 replaced by θ,

$$\begin{aligned} e'(|x'| - t',\theta',\theta_0') &= e'(x'\cdot\theta' - t',\theta',\theta_0') \\ &= e(x\cdot\theta - t,\theta,\theta_0) = e(|x| - t,\theta,\theta_0) \end{aligned} \tag{5.11}$$

where

$$e(\tau,\theta,\theta_0) = e'(\gamma^{-1}\tau,\theta',\theta_0') \tag{5.12}$$

$$\gamma = \frac{1-v\theta^1}{\sqrt{1-v^2}} = \frac{1-\vec{v}\cdot\theta}{\sqrt{1-v^2}} \tag{5.13}$$

and θ_0' (respectively θ') are derived from θ_0 (respectively θ) by (5.6). Thus the echo is described relative to the sonar system frame by

$$u_{sc}^{\infty}(t,x) = \frac{e(|x|-t,\theta,\theta_0)}{|x'|} \tag{5.14}$$

where the echo waveform $e(\tau,\theta,\theta_0)$ is defined by (5.10), (5.12) and (5.13). The resulting form of $e(\tau,\theta,\theta_0)$ can be written

$$e(\tau,\theta,\theta_0) = \frac{\gamma}{\gamma_0} \operatorname{Re}\left\langle \left\{ \int_0^{\infty} e^{i\tau\omega}\, \hat{s}\left(\frac{\gamma}{\gamma_0}\,\omega\right) T_+(\omega\gamma\theta',\omega\gamma\theta_0')\,d\omega \right\} \right\rangle \tag{5.15}$$

If $v = 0$ then $\gamma = \gamma_0 = 1$, $\theta' = \theta$, $\theta_0' = \theta_0$ and (5.15) reduces to equation (4.10) for the echo waveform of a stationary scatterer. Comparison of (4.10) and (5.15) shows that the motion of the scatterer relative to the sonar system causes three kinds of distortion in the echo. These are a frequency shift in the signal waveform of amount

$$\Delta\omega = \frac{\gamma}{\gamma_0} - 1, \tag{5.16}$$

a frequency shift in the scattering amplitude T_+ of amount

$$\Delta'\omega = \gamma - 1 \tag{5.17}$$

and an angular distortion in the scattering cross section T_+ due to the replacement of θ and θ_0 by θ' and θ_0'. The frequency shift (5.16) is the usual Doppler shift.

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